

Wave function collapse implies divergence of average displacement

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We show that propagating a truncated discontinuous wave function by Schrödinger's equation, as asserted by the collapse axiom, gives rise to non-existence of the average displacement of the particle on the line. It also implies that there is no Zeno effect. On the other hand, if the truncation is done so that the reduced wave function is continuous, the average coordinate is finite and there is a Zeno effect. Therefore the collapse axiom of measurement needs to be revised.

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I. INTRODUCTION

We consider the following instantaneous measurement of the location of a free quantum particle in one dimension [1]. At a given instant the entire line, excluding a finite interval, is illuminated to see if the particle is in the illuminated region. If the result of the measurement is negative, that is, if the particle is not observed in illuminated region, the collapse axiom of quantum mechanics (the collapse or reduction axiom) asserts that the wave function after the measurement is the left limit (in time) of the wave function at the instant of measurement, truncated outside the interval and renormalized. Alternatively, if the illuminated region is a finite interval and the measurement establishes that the particle is in the illuminated interval, without giving its exact location, the same axiom makes the same assertion as above [2]. In this paper we consider the propagation of the wave function after the reduction. The initial condition for the propagation is discontinuous, specifically, it has jump discontinuities at the endpoints of the interval and vanishes identically in the illuminated region.

We show that propagating the discontinuous wave function by Schrödinger's equation gives rise to non-existence of the average displacement of the particle on the line or half the line. One consequence of this phenomenon is that the speed of the average particle displacement on a given interval is arbitrarily large if the interval is sufficiently large. In particular, it can exceed the speed of light. Therefore the collapse axiom of measurement needs to be revised. On the other hand, if the truncation is done so that the reduced wave function is continuous, the average coordinate is finite.

II. AN EXACTLY SOLVABLE EXAMPLE

To fix the ideas, we consider first the propagation of the initial rectangle

$$\psi_0(x) = \begin{cases} 1 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise,} \end{cases}$$

which is typical of the propagation of more general initial conditions with jump discontinuities. The propagated wave function is given by

$$\Psi(y, t) = \sqrt{\frac{m}{2\pi i \hbar t}} \int_0^1 \exp\left\{\frac{im(x-y)^2}{2\hbar t}\right\} dx, \quad (1)$$

so its decay for $y \rightarrow \infty$ is determined by the asymptotic behavior of the function $\text{erfc}(y)$ in the complex plane. It is given by (see [3])

$$\begin{aligned} \Psi(y, t) &= \frac{1}{\pi} \sqrt{\frac{2\hbar t}{im}} \\ &\times \left[\frac{\exp\left\{-\frac{imy^2}{2\hbar t}\right\}}{y} - \frac{\exp\left\{-\frac{im(y-1)^2}{2\hbar t}\right\}}{y-1} \right] + O\left(\frac{1}{y^3}\right) \end{aligned}$$

as $y \rightarrow \infty$. It follows that

$$|\Psi(y, t)|^2 = O\left(\frac{1}{y^2}\right) \quad \text{as } y \rightarrow \infty, \quad (2)$$

hence

$$\int_0^\infty y |\Psi(y, t)|^2 dy = \infty. \quad (3)$$

Although the function $|\Psi(y, t)|^2$ is integrable, the function $y |\Psi(y, t)|^2$ is not absolutely integrable on the line. Therefore the average $\langle x(t) \rangle$ does not exist.

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III. PROPAGATION OF A POLYGON

We assume that the negative measurement is done outside the finite interval $[-a, 0]$ at time $t = 0$, so that wave function is truncated outside the interval. We consider two cases: (i) the truncated wave function is discontinuous, and (ii) the truncated wave function is continuous, though its derivative may be discontinuous. To determine the wave function at a short time $\Delta t > 0$, we approximate the truncated wave function by an inscribed polygon.

To evaluate the propagated wave function, we divide the interval $[-a, 0]$ into N subintervals and approximate the function $\Psi_0(x)$ in the j -th subinterval by the linear function $a_j x + b_j$. The free propagation of the approximating function in each subinterval is given by the integral

$$\psi_j(y, \Delta t) = \sqrt{\frac{m}{2\pi i \hbar \Delta t}} \int_{-a+j\Delta x}^{-a+(j+1)\Delta x} (a_j x + b_j) \exp\left\{\frac{im(x-y)^2}{2\hbar\Delta t}\right\} dx. \quad (4)$$

This integral is a sum of two integrals of the form

$$I_0^j = b_j \sqrt{\frac{m}{2\pi i \hbar \Delta t}} \int_A^B \exp\left\{\frac{im(x-y)^2}{2\hbar\Delta t}\right\} dx \quad (5)$$

and

$$I_1^j = a_j \sqrt{\frac{m}{2\pi i \hbar \Delta t}} \int_A^B x \exp\left\{\frac{im(x-y)^2}{2\hbar\Delta t}\right\} dx. \quad (6)$$

Changing the variable of integration in eq.(5) to $x-y = u$ and then $(A-y)z = u$ in the first and $(B-x)z = u$ in the second integral gives

$$I_0^j = b_j \sqrt{\frac{m}{2\pi i \hbar \Delta t}} \left[-(A-y) \int_0^1 \exp\left\{\frac{im(A-y)^2 z^2}{2\hbar\Delta t}\right\} dz + (B-y) \int_0^1 \exp\left\{\frac{im(B-y)^2 z^2}{2\hbar\Delta t}\right\} dz \right]. \quad (7)$$

Each one of the two integrals can be evaluated by the asymptotic formula [4]

$$\int_0^1 \exp\{ixt^2\} dt \sim \frac{1}{2} \sqrt{\frac{\pi}{x}} e^{i\pi/4} - \frac{i}{2} e^{ix} \sum_{n=0}^{\infty} (-i)^n \frac{\Gamma\left(n + \frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right) x^{n+1}}, \quad x \gg 1. \quad (8)$$

Applying (8) to the first integral in eq.(7) with $x = \frac{m(A-y)^2}{2\hbar\Delta t}$ and to the second integral with $x = \frac{m(B-y)^2}{2\hbar\Delta t}$, gives

$$I_0^j = b_j \sqrt{\frac{i\Delta t \hbar}{2m\pi}} \left[\frac{\exp\left\{\frac{im(A-y)^2}{2\hbar\Delta t}\right\}}{A-y} - \frac{\exp\left\{\frac{im(B-y)^2}{2\hbar\Delta t}\right\}}{B-y} \right] + O\left(\frac{\Delta t^{3/2}}{(A-y)^3}\right) + O\left(\frac{\Delta t^{3/2}}{(B-y)^3}\right). \quad (9)$$

This expansion is valid for small Δt and y away from A and B (see below).

Next, from eq.(6), we have

$$\begin{aligned} I_1^j &= a_j \sqrt{\frac{m}{2\pi i \hbar \Delta t}} \left[y \int_A^B \exp\left\{\frac{im(x-y)^2}{2\hbar\Delta t}\right\} dx + \int_A^B (x-y) \exp\left\{\frac{im(x-y)^2}{2\hbar\Delta t}\right\} dx \right] \\ &= I + II. \end{aligned}$$

The first integral, I , is similar to I_0^j and is given by

$$\begin{aligned} I &= a_j y \sqrt{\frac{i\Delta t \hbar}{2m\pi}} \\ &\times \left[\frac{\exp\left\{\frac{im(A-y)^2}{2\hbar\Delta t}\right\}}{A-y} - \frac{\exp\left\{\frac{im(B-y)^2}{2\hbar\Delta t}\right\}}{B-y} \right] \\ &+ O\left(\frac{\Delta t^{3/2}}{(A-y)^3}\right) + O\left(\frac{\Delta t^{3/2}}{(B-y)^3}\right) \end{aligned} \quad (10)$$

(see eq.(9)). In the second integral, II , we set $u = x - y$

and obtain

$$\begin{aligned} II &= a_j \sqrt{\frac{m}{2\pi i \hbar \Delta t}} \int_{A-y}^{B-y} u \exp \left\{ \frac{im u^2}{2\hbar \Delta t} \right\} du \\ &= -a_j \sqrt{\frac{i \hbar \Delta t}{2m\pi}} \\ &\quad \left[\exp \left\{ \frac{im(B-y)^2}{2\hbar \Delta t} \right\} - \exp \left\{ \frac{im(A-y)^2}{2\hbar \Delta t} \right\} \right]. \end{aligned} \quad (11)$$

Now, using eqs.(10) and (11), we obtain

$$\begin{aligned} I_1^j &= I + II = a_j \sqrt{\frac{i \hbar \Delta t}{2m\pi}} \\ &\quad \left[\frac{A \exp \left\{ \frac{im(A-y)^2}{2\hbar \Delta t} \right\}}{A-y} - \frac{B \exp \left\{ \frac{im(B-y)^2}{2\hbar \Delta t} \right\}}{B-y} \right] \\ &\quad + O \left(\frac{\Delta t^{3/2}}{(A-y)^3} \right) + O \left(\frac{\Delta t^{3/2}}{(B-y)^3} \right), \end{aligned} \quad (12)$$

For $B = 0$ and positive y , we obtain

$$\begin{aligned} I_1^j(B=0) &= a_j \sqrt{\frac{i \hbar \Delta t}{2m\pi}} \frac{A \exp \left\{ \frac{im(A-y)^2}{2\hbar \Delta t} \right\}}{A-y} \\ &\quad + O \left(\frac{\Delta t^{3/2}}{(A-y)^3} \right) + O \left(\frac{\Delta t^{3/2}}{y^3} \right). \end{aligned}$$

To calculate the propagation of the entire polygon, we choose the N vertices at the points

$$x_j = -a + j\Delta x, \quad \Delta x = \frac{a}{N}.$$

Each side of the polygon is a line $y = a_j x + b_j$, with

$$\begin{aligned} a_j &= \frac{\psi(x_{j+1}, t) - \psi(x_j, t)}{\Delta x} \\ b_j &= \psi(x_j, t) - \frac{\psi(x_{j+1}, t) - \psi(x_j, t)}{\Delta x} x_j. \end{aligned} \quad (13)$$

Note that

$$x_N = 0, \quad x_{N-1} = -\Delta x, \quad b_0 = a_0 a, \quad b_{N-1} = 0. \quad (14)$$

The contribution of the interval $[x_j, x_{j+1}]$ to the integral (4) is given by

$$\begin{aligned} \psi_j(y, \Delta t) &= \sqrt{\frac{m}{2\pi i \hbar \Delta t}} \\ &\quad \int_{-a+j\Delta x}^{-a+(j+1)\Delta x} (a_j x + b_j) \exp \left\{ \frac{im(x-y)^2}{2\hbar \Delta t} \right\} dx. \end{aligned}$$

We write this as

$$\begin{aligned} \psi_j(y, \Delta t) &= \sqrt{\frac{m}{2\pi i \hbar \Delta t}} \left[a_j \int_{-a+j\Delta x}^0 x \exp \left\{ \frac{im(x-y)^2}{2\hbar \Delta t} \right\} dx \right. \\ &\quad \left. - a_j \int_{-a+(j+1)\Delta x}^0 x \exp \left\{ \frac{im(x-y)^2}{2\hbar \Delta t} \right\} dx + b_j \int_{-a+j\Delta x}^{-a+(j+1)\Delta x} \exp \left\{ \frac{im(x-y)^2}{2\hbar \Delta t} \right\} dx \right] \end{aligned} \quad (15)$$

for all $j < N$ and fixed $y > 0$. Using the integrals (9) and (12) in (15), we obtain

$$\begin{aligned} \psi_j(y, \Delta t) &\sim \sqrt{\frac{i \hbar \Delta t}{2m\pi}} \\ &\quad \times \left\{ a_j \left[\frac{(a-j\Delta x) \exp \left\{ \frac{im(-a+j\Delta x-y)^2}{2\hbar \Delta t} \right\}}{-a+j\Delta x-y} - \frac{(a-(j+1)\Delta x) \exp \left\{ \frac{im(-a+(j+1)\Delta x-y)^2}{2\hbar \Delta t} \right\}}{-a+(j+1)\Delta x-y} \right] \right. \\ &\quad \left. + b_j \left[\frac{\exp \left\{ \frac{im(-a+j\Delta x-y)^2}{2\hbar \Delta t} \right\}}{-a+j\Delta x-y} - \frac{\exp \left\{ \frac{im(-a+(j+1)\Delta x-y)^2}{2\hbar \Delta t} \right\}}{-a+(j+1)\Delta x-y} \right] \right\} = \sqrt{\frac{i \hbar \Delta t}{2m\pi}} [a_j S_j + b_j R_j]. \end{aligned} \quad (16)$$

First, we define $\alpha = \frac{m}{2\hbar\Delta t}$, and using the abbreviation $x_j = -a + j\Delta x$, we rewrite eq.(16) as

$$\begin{aligned} \psi_j(y, \Delta t) \sim & \sqrt{\frac{i}{4\alpha\pi}} \left\{ a_j \left[\frac{x_j \exp\{i\alpha(x_j - y)^2\}}{x_j - y} - \frac{x_{j+1} \exp\{i\alpha(x_{j+1} - y)^2\}}{x_{j+1} - y} \right] \right. \\ & \left. + b_j \left[\frac{\exp\{i\alpha(x_j - y)^2\}}{x_j - y} - \frac{\exp\{i\alpha(x_{j+1} - y)^2\}}{x_{j+1} - y} \right] \right\} \end{aligned}$$

or

$$\psi_j(y, \Delta t) \sim \sqrt{\frac{i}{4\alpha\pi}} \left\{ (a_j x_j + b_j) \frac{\exp\{i\alpha(x_j - y)^2\}}{x_j - y} - (a_j x_{j+1} + b_j) \frac{\exp\{i\alpha(x_{j+1} - y)^2\}}{x_{j+1} - y} \right\}.$$

The propagated polygon is

$$S = \sqrt{\frac{i}{4\alpha\pi}} \sum_{j=0}^{N-1} \left\{ (a_j x_j + b_j) \frac{\exp\{i\alpha(x_j - y)^2\}}{x_j - y} - (a_{j+1} x_{j+1} + b_{j+1}) \frac{\exp\{i\alpha(x_{j+1} - y)^2\}}{x_{j+1} - y} \right\} + O\left(\frac{\Delta t^{3/2}}{y^3}\right)$$

for fixed $y < -a$ and $y > 0$. Using the identity

$$a_j x_j + b_j = a_{j-1} x_j + b_{j-1} \quad (17)$$

(that expresses the continuity of the polygon at the vertices), we find that the sum is telescopic and reduces to

$$S = \sqrt{\frac{i}{4\alpha\pi}} (a_0 x_0 + b_0) \frac{\exp\{i\alpha(x_0 - y)^2\}}{x_0 - y} - \sqrt{\frac{i}{4\alpha\pi}} (a_N x_N + b_N) \frac{\exp\{i\alpha(x_N - y)^2\}}{x_N - y} + O\left(\frac{\Delta t^{3/2}}{y^3}\right),$$

hence

$$\Psi(y, t) = \sqrt{\frac{i}{4\alpha\pi}} \left[\frac{\Psi(-a, 0) \exp\{i\alpha(x_0 - y)^2\}}{x_0 - y} - \frac{\Psi(0, 0) \exp\{i\alpha(x_N - y)^2\}}{x_N - y} \right] + O\left(\frac{\Delta t^{3/2}}{y^3}\right).$$

Thus

$$|\Psi(y, t)|^2 = \frac{1}{4\alpha\pi} \left\{ \frac{|\Psi(-a, 0)|^2}{(x_0 - y)^2} + \frac{|\Psi(0, 0)|^2}{(x_N - y)^2} - 2\Re \left[e^{i\alpha[(x_N - y)^2 - (x_0 - y)^2]} \frac{\Psi^*(-a, 0)\Psi(0, 0)}{(x_0 - y)(x_N - y)} \right] \right\} + O\left(\frac{\Delta t^2}{y^4}\right).$$

That is, if $\Psi(-a, 0) \neq 0$ or $\Psi(0, 0) \neq 0$, then

$$|\Psi(y, t)|^2 = O\left(\frac{\Delta t}{y^2}\right) \quad \text{uniformly for } y < -a - \delta, y > \delta \quad (18)$$

for some $\delta > 0$.

Thus there is propagation in short (and long) time and the particle can be found at any point. As in Section 2, $\langle x(\Delta t) \rangle$ does not exist. In particular, the average on an

interval $[-a, b]$ at time Δt can be made arbitrarily large if b is chosen sufficiently large. This means that the speed of the average on a sufficiently large interval can exceed the speed of light (see [5]). Equation (18) also means that there is propagation in short time, so there is no Zeno effect for continuous time measurements, in contrast to the assertion of existing theories [6].

If, however, $\Psi(y, 0)$ is continuous, that is, $\Psi(-a, 0) =$

$\Psi(0, 0) = 0$, the above analysis gives

$$|\Psi(y, t)|^2 = O\left(\frac{\Delta t^3}{y^6}\right) \quad \text{uniformly for } y < -a - \delta, y > \delta \quad (19)$$

for some $\delta > 0$. This means that there is no propagation

in short time and there is the Zeno effect. In this case $\langle x(t) \rangle$ is finite.

Note that in case the Hamiltonian contains a finite potential $V(x)$, the short time propagation, given in [7], [8]

$$\Psi(y, t + \Delta t) = \sqrt{\frac{m}{2\pi i \hbar t}} \int_{-\infty}^{\infty} \Psi(x, t) \exp \left\{ \frac{im(x-y)^2}{2\hbar \Delta t} + i\hbar \Delta t V(x) \right\} dx (1 + O(\Delta t)),$$

gives the same-short time result as above. That is, the discontinuity of the wave function leads to an infinite average. Note also that the average momentum also diverges.

IV. SUMMARY AND DISCUSSION

When the mean displacement is infinite the sample average of measurements of single particle measurements of displacement will not converge as the sample size increases. This situation is unacceptable in quantum mechanics, which interprets measurements through moments (e.g., the uncertainty principle). In particular, the Ehrenfest theorem does not hold in this case.

There seems to be no natural cutoff that allays this situation, which leads to the conclusion that the collapse axiom of measurements leads to unphysical results. On the other hand, as the above calculations indicate, a

measurement axiom that truncates the pre-measurement wave function to a continuous post-measurement wave function circumvents the above mentioned difficulty. The choice of a continuous post-measurement wave function is an open problem, which will be discussed separately.

The evolution of a particle, initially distributed uniformly in a finite potential well, was discussed in [9]. Its wave function develops a fractal space-time structure. In the case of continuous measurements inside the infinite well one gets the quantum Zeno dynamics [10]. According to our result, there is no Zeno effect if measurements are negative outside an interval.

Another consequence of the infinite post-measurement average of the displacement is that the average can move faster than the speed of light, which is a new violation of special relativity. This is also a new violation of quantum information theory, because the proof that information cannot travel faster than light concerns finite sets of discrete degrees of freedom, such as spins [5], [11].

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